

Discriminants of multilinear systems

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Abstract

We study well-constrained bilinear algebraic systems in order to formulate their discriminant. We derive a new determinantal formula for the discriminant of a multilinear system that appears in the study of Nash equilibria of multiplayer games with mixed strategies.

1 Introduction

We study well-constrained bilinear algebraic systems. We aim at compact formulae for the discriminant of such systems so as to improve the complexity of computing them. One method would be for discriminants to be computed via implicitization [EKK⁺13].

In general, matrix formulae for the discriminant would be preferable but they are quite hard to obtain and very few currently exist. For instance, the discriminant of a single univariate polynomial is given, up to a multiplicative monomial factor, by the determinant of the Sylvester matrix of the polynomial and its derivative. For a more general study see [Stu16].

The lack of compact discriminant formulae is in contrast to resultant matrices, which have been extensively studied and for which compact formulae exist for a large number of system families. In particular, the resultant matrices of overconstrained multihomogeneous systems have been studied by Dickenstein, Mantzaflaris, and Emiris [DE03, EM12] and, earlier, by Sturmfels, Weyman, and Zelevinsky [SZ94, WZ94].

2 Purely bilinear systems

Consider a bilinear polynomial system of $n + m$ equations on $\mathbb{R}^n \times \mathbb{R}^m$:

$$F_k : \sum_{i=0}^n \sum_{j=0}^m a_{i,j}^{(k)} x_i y_j = 0, \quad 1 \leq k \leq n + m. \quad (1)$$

The set of monomials appearing in each polynomial is generically $A = \{x_0, x_1, \dots, x_n\} \times \{y_0, y_1, \dots, y_m\}$. Assuming the system is unmixed, the generic number of solutions is the volume (normalized to 1 for unit simplex Δ_{n+m}) of the simplex product $\Delta_n \times \Delta_m$:

$$\binom{n+m}{n} = (n+m)! \left(\frac{1}{n!} \times \frac{1}{m!} \right). \quad (2)$$

The *discriminant* $\Delta_A(F_1, \dots, F_{n+m})$ of the system is the irreducible polynomial (with coprime coefficients, defined up to a sign) in the coefficients $a_{i,j}^{(k)}$ which vanishes whenever the system (1) has a multiple solution.

The discriminant can be computed (up to superfluous factors) by eliminating all affine variables except one, and computing the discriminant of the univariate elimination polynomial. For $n = m = 1$, with $a_{ij} = a_{i,j}^{(1)}$,

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$b_{ij} = a_{i,j}^{(2)}$, we have

$$\Delta_A(F_1, F_2) = \left(\begin{vmatrix} a_{00} & a_{01} \\ b_{10} & b_{11} \end{vmatrix} - \begin{vmatrix} a_{10} & a_{11} \\ b_{00} & b_{01} \end{vmatrix} \right) \left(\begin{vmatrix} a_{00} & a_{10} \\ b_{01} & b_{11} \end{vmatrix} - \begin{vmatrix} a_{01} & a_{11} \\ b_{00} & b_{10} \end{vmatrix} \right) - 4 \begin{vmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{vmatrix} \begin{vmatrix} b_{00} & b_{01} \\ b_{10} & b_{11} \end{vmatrix}. \quad (3)$$

3 Degree bound

This section bounds the degree of the discriminant.

For sparse polynomial systems, a degree bound was obtained by Cattani, Cueto, Dickenstein, di Rocco and Sturmfels [CCD⁺14] and a more special case settled in [DEK14].

The discriminant equals the resultant of the equations (1) and $J = 0$, where

$$J = \det \left(\begin{array}{cc} \frac{\partial F_i}{\partial x_j} \Big|_{j=1 \dots n} & \frac{\partial F_i}{\partial y_j} \Big|_{j=1 \dots m} \end{array} \right)_{i=1 \dots n+m} \quad (4)$$

is the Jacobian (determinant). The first n columns of the Jacobian matrix do not depend on the variables x_j and are linear in the variables y_j , while the last m columns of the Jacobian matrix do not depend on the variables y_j and are linear in the variables x_j . Therefore the Jacobian is (homogeneous) of degree m in the x_j 's, and of degree n in the y_j 's. The support is a product of scaled simplexes: $m\Delta_n \times n\Delta_m$. The Jacobian is multilinear in the coefficients of $a_{i,j}^{(k)}$, linear for each group with fixed k .

To bound the degree of the discriminant in the variables $a_{i,j}^{(1)}$, we compute

$$MV(J, F_2, \dots, F_{n+m}) + MV(F_1, F_2, \dots, F_{n+m}) \deg_k J.$$

The first term is (up to the factor $1/n!m!$) the permanent of a $(n+m) \times (n+m)$ matrix with m n 's and n m 's in one row, and all other entries equal to 1, hence $2nm(n+m-1)!/n!m!$. The degree bound is

$$\left(\frac{2nm}{n+m} + 1 \right) \binom{n+m}{n}. \quad (5)$$

The total degree is $n+m$ times larger.

The actual degrees appear to be smaller: 2 instead of 4 for $n = m = 1$, and 4 instead of 7 for $\{n, m\} = \{1, 2\}$.

4 Ideals containing the discriminant

Theorem 4.1. *The discriminant of the bilinear system (1) is in the ideal generated by the maximal minors of the $(m+n)(n+1) \times (m+1)$ matrix*

$$(\partial F_i / \partial x_j)_{1 \leq i \leq m+n, 0 \leq j \leq n}, \quad (6)$$

where each row represents a linear polynomial in the y_k 's, and the columns correspond to the variables $(y_0 : y_1 : \dots : y_m)$.

Proof. We have to prove that if the maximal minors vanish, the discriminant is zero. If the maximal minors vanish, we have a kernel (u_0, u_1, \dots, u_m) of the matrix in (6). Hence the $(m+n)(n+1)$ derivatives $\partial F_i / \partial x_j$ vanish with all $y_i = u_i$. By Euler's relation

$$F_i = \sum_{k=0}^n x_k \frac{\partial F_i}{\partial x_k} \quad (7)$$

we conclude that all F_i vanish at all $y_i = u_i$ and with any x_i . The derivatives $\partial F_i / \partial x_j$ with $j \neq 0$ form the $(m+n) \times n$ zero submatrix of the Jacobian. Thus we have a whole subspace of singular solutions of the system (1), hence the discriminant vanishes. \square

Similarly, the discriminant must be in the ideal generated by the minors of the $(m+n)(m+1) \times (n+1)$ matrix

$$(\partial F_i / \partial y_j)_{1 \leq i \leq m+n, 0 \leq j \leq m},$$

where the columns correspond to the variables $(x_0 : x_1 : \dots : x_m)$. In particular, the discriminant (3) is in the minor ideals I_1, I_2 of

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \\ b_{00} & b_{01} \\ b_{10} & b_{11} \end{pmatrix}, \quad \begin{pmatrix} a_{00} & a_{10} \\ a_{01} & a_{11} \\ b_{00} & b_{10} \\ b_{01} & b_{11} \end{pmatrix}. \quad (8)$$

The discriminant is then in the radical of the product ideal of (intersecting) I_1 and I_2 . In this case, the discriminant is in the product ideal itself.

Other ideals are: *higher discriminant* ideals, characterizing the parameters of the polynomial system with a multiple root of higher multiplicity, or more than one multiple root. Also, the ideal defining the singularity locus of the discriminant hypersurface.

5 Sparse systems

This section focuses on sparse multilinear systems, in particular when each polynomial (or subset of polynomials) does not depend on a subset of the variables. These appear in the study of Nash equilibria in [EV14].

Consider the bilinear system on $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$:

$$\begin{aligned} H_1 : & \quad a_0 x_1 y_1 + a_1 x_1 y_0 + a_2 x_0 y_1 + a_4 x_0 y_0 = 0, \\ H_2 : & \quad b_0 x_1 z_1 + b_1 x_1 z_0 + b_3 x_0 z_1 + b_4 x_0 z_0 = 0, \\ H_3 : & \quad c_0 y_1 z_1 + c_2 y_1 z_0 + c_3 y_0 z_1 + c_4 y_0 z_0 = 0. \end{aligned} \quad (9)$$

The generic number of solutions equals 2. The discriminant equals

$$\begin{aligned} \Delta(H_1, H_2, H_3) = & \left(a_0 \begin{vmatrix} b_3 & b_4 \\ c_3 & c_4 \end{vmatrix} - a_1 \begin{vmatrix} b_3 & b_4 \\ c_0 & c_2 \end{vmatrix} - a_2 \begin{vmatrix} b_0 & b_1 \\ c_3 & c_4 \end{vmatrix} + a_4 \begin{vmatrix} b_0 & b_1 \\ c_0 & c_2 \end{vmatrix} \right)^2 \\ & - 4 \begin{vmatrix} a_0 & a_1 \\ a_2 & a_4 \end{vmatrix} \begin{vmatrix} b_0 & b_1 \\ b_3 & b_4 \end{vmatrix} \begin{vmatrix} c_0 & c_2 \\ c_3 & c_4 \end{vmatrix}. \end{aligned} \quad (10)$$

In the following theorem, the 6×6 matrix is made up of column pairs similar to the 4×2 matrices in (8): the first two columns encode all partial derivatives of H_1, H_2, H_3 that are linear in x_1, x_0 , etc. The columns of the 6×6 matrix thereby correspond to the multihomogeneous variables $(x_1 : x_0), (y_1 : y_0), (z_1 : z_0)$.

Theorem 5.1. *The discriminant $\Delta(H_1, H_2, H_3)$ equals*

$$\det \begin{pmatrix} 0 & 0 & a_0 & a_1 & b_0 & b_1 \\ 0 & 0 & a_2 & a_4 & b_3 & b_4 \\ a_0 & a_2 & 0 & 0 & c_0 & c_2 \\ a_1 & a_4 & 0 & 0 & c_3 & c_4 \\ b_0 & b_3 & c_0 & c_3 & 0 & 0 \\ b_1 & b_4 & c_2 & c_4 & 0 & 0 \end{pmatrix}. \quad (11)$$

Proof. For a conceptual proof, we relate a multiple root $(x_1 : x_0), (y_1 : y_0), (z_1 : z_0)$ of the system (9) and the kernel $(\lambda_1 : \lambda_2 : \lambda_3)$ of the transposed Jacobian

$$\begin{pmatrix} \partial F_1 / \partial x_1 & \partial F_2 / \partial x_1 & 0 \\ \partial F_1 / \partial y_1 & 0 & \partial F_3 / \partial y_1 \\ 0 & \partial F_2 / \partial z_1 & \partial F_3 / \partial z_1 \end{pmatrix} \quad (12)$$

to a kernel vector $(u_1 : u_2 : u_3 : u_4 : u_5 : u_6)$ of the matrix in (11), and vice versa. The relation is as follows:

$$(u_1 : u_2 : u_3 : u_4 : u_5 : u_6) = \left(\frac{x_1}{\lambda_3} : \frac{x_0}{\lambda_3} : \frac{y_1}{\lambda_2} : \frac{y_0}{\lambda_2} : \frac{z_1}{\lambda_1} : \frac{z_0}{\lambda_1} \right),$$

$$\left(\frac{x_1}{x_0}, \frac{y_1}{y_0}, \frac{z_1}{z_0} \right) = \left(\frac{u_1}{u_2}, \frac{u_3}{u_4}, \frac{u_5}{u_6} \right),$$

The 1st, 3rd and 5th rows of (11) multiplied by the vector $(x_1, x_0, y_1, y_0, z_1, z_0)$ give the non-zero entries of the Jacobian matrix. The 2nd, 4th and 6th rows give the derivatives with respect to x_0, y_0, z_0 . This allows to complete three Euler identities like (7), and relate the kernel element with a singular root of (9). \square

We have the following observation.

Lemma 5.2. *The system (9) has a multiple root if and only if the quadratic form $F+G+H$ (in the six variables $x_1, x_0, y_1, y_0, z_1, z_0$) degenerates.*

The 2×2 blocks of (11) represent the following derivatives:

$$\begin{pmatrix} 0 & \partial F_1/\partial x & \partial F_2/\partial x \\ \partial F_1/\partial y & 0 & \partial F_3/\partial y \\ \partial F_2/\partial z & \partial F_3/\partial z & 0 \end{pmatrix}. \quad (13)$$

The determinants of the matrices (13) and (12) match formally.

Direct generalization to $\mathbb{P}^k \times \mathbb{P}^\ell \times \mathbb{P}^m$ is hardly possible if the equation blocks have the sizes k, ℓ, m , because the derivative blocks have non-matching number of columns. But we might assume the equation blocks to be of equal size, and then the matrix is constructed correctly. But would its determinant indeed be the system discriminant?

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